Fano Manifolds, Contact Structures, and Quaternionic Geometry

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Abstract

Let Z be a compact complex $(2n+1)$-manifold which carries a *complex contact structure*, meaning a codimension-1 holomorphic sub-bundle $D \subset TZ$ which is maximally non-integrable. If Z admits a Kähler-Einstein metric of positive scalar curvature, we show that it is the Salamon twistor space of a quaternion-Kähler manifold (M^{4n}, g). If Z also admits a second complex contact structure $\tilde{D} \neq D$, then $Z = \mathbf{CP}_{2n+1}$. As an application, we give several new characterizations of the Riemannian manifold $\mathbf{HP}_n = Sp(n+1)/(Sp(n) \times Sp(1))$.

1 Introduction

If (M, g) is an oriented Riemannian m-manifold, how many tensor fields $\varphi \neq 0$ does M admit which satisfy $\nabla\varphi = 0$, where ∇ is the Levi-Cività connection associated with the Riemannian metric g? Obviously there are always some such fields: the metric, the volume form, their inverses, and linear combinations of contracted tensor products of these. There will be more only if the geometry of M is special in the sense that the *holonomy group* — that is [8, 26], the group of linear transformation of a tangent space induced by parallel transport around loops — is a proper subgroup of $SO(m)$, and a fundamental research topic for present-day Riemannian geometry is the problem of classifying those Riemannian manifolds whose holonomy geometries are special in this sense. In fact, if we exclude the locally symmetric spaces (for which the curvature tensor $\mathcal{R}$ satisfies $\nabla\mathcal{R} = 0$) and local products of manifolds of lower dimension, there are (up to conjugation) only seven possible families of connected holonomy groups: $SO(m)$, $U(m/2)$, $SU(m/2)$, G_2 $(m = 7)$, $\mathrm{Spin}(7)$ $(m = 8)$, $Sp(m/4)$, and

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$[Sp(m/4) \times Sp(1)]/\mathbf{Z}_2$ ($m \geq 8$). The present paper is largely motivated by the problem of understanding the last of these holonomy families.

Definition 1.1 *A Riemannian manifold (M, g) of dimension $4n$, $n \geq 2$, is said to be quaternion-Kähler if its holonomy group is (conjugate to) a subgroup of $[Sp(n) \times Sp(1)]/\mathbf{Z}_2 \subset SO(4n)$.*

This terminology may be justified by the fact [8, 26] that any such manifold carries a 4-form φ such that $\nabla\varphi = 0$, and this 4-form may be thought of as a quaternionic analogue of the 2-form which plays such a central rôle in the Kähler geometry of complex manifolds.

A quaternion-Kähler manifold is necessarily Einstein [7], and its scalar curvature is non-zero iff the holonomy group contains the $Sp(1)$ factor of $[Sp(n) \times Sp(1)]/\mathbf{Z}_2$. In particular, such a manifold is necessarily compact provided it is complete and its (constant) scalar curvature is positive.

Definition 1.2 *A quaternion-Kähler manifold (M, g) will be called* positive *if it is compact and has positive scalar curvature.*

Of course, multiplying the metric g of such a manifold by a positive constant always gives us 'new' examples in a completely trivial way. We henceforth eliminate this possibility by normalizing the scalar curvature of g to be the same as that of the standard metric on $\mathbf{HP}_n = S^{4n+3}/Sp(1)$, namely $s = 16n(n+2)$.

It should be emphasized that a quaternion-Kähler manifold is typically *not* a Kähler manifold; $[Sp(n) \times Sp(1)]/\mathbf{Z}_2 \not\subset U(2n)$. Nonetheless, questions about such manifolds can, in the positive case, be reduced to problems in algebraic geometry over $\mathbf{C}$ by means of a *twistor construction* [24, 6] which associates a compact complex $(2n+1)$-manifold Z with any positive quaternion-Kähler $4n$-manifold (M, g). Moreover, the so-called twistor spaces Z that arise by this construction have some remarkable properties. First, they admit Kähler-Einstein metrics of positive scalar curvature; in particular, they have $c_1 > 0$, and so are *Fano manifolds*. Secondly, any such Z admits a *complex contact structure*, meaning a maximally non-integrable holomorphic sub-bundle $D \subset TZ$ of complex codimension 1.

In short, any twistor space Z is a *Fano contact manifold* in the sense of Definition 2.2 below. On the other hand, every known Fano contact manifold is actually a twistor space. Our first main result indicates that this is no accident:

Theorem A *Let Z be a Fano contact manifold. Then Z is a twistor space iff it admits a Kähler-Einstein metric.*

Based on this, it might seem reasonable to conjecture that every Fano contact manifold is a twistor space. For other evidence in favor of such a conjecture, see [17, 18, 23, 30].

The same machinery used to prove the above also allows us to address the problem of determining which Fano contact manifolds carry more than one contact structure:

Theorem B *Let Z be a Fano manifold of complex dimension $2n+1$ which admits a Kähler-Einstein metric. If Z admits two distinct complex contact structure $D, \tilde{D} \subset TZ$, then $Z \cong \mathbf{CP}_{2n+1}$.*

This result has some interesting ramifications regarding quaternion-Kähler manifolds. To give a typical example, let us say that a self-diffeomorphism $\Phi : M \to M$ of a quaternion-Kähler manifold (M, g) is a *quaternionic automorphism* if it preserves the $[GL(n, \mathbf{H}) \times Sp(1)]/\mathbf{Z}_2$ structure determined by g; thus the projective transformations of $\mathbf{HP}_n$ induced by elements of $GL(n+1, \mathbf{H})$ are quaternionic automorphism of $\mathbf{HP}_n$, whereas only elements of the subgroup $Sp(n+1) \times \mathbf{R}^+ \subset GL(n+1, \mathbf{H})$ act by isometries. However, Theorem B implies that this example is essentially anomalous:

Corollary C *Let (M, g) be a positive quaternion-Kähler $4n$-manifold. If there is a quaternionic automorphism $\Phi : M \to M$ which does not preserve g, then (M, g) is isometric to the symmetric space $\mathbf{HP}_n$.*

Notation. Certain notational conventions employed in these pages may elicit gasps from the occasional algebraic geometer who takes the time to peruse them. Thus multiplicative rather than additive notation will be employed for the Picard group, and our convention regarding projectivizations of vector spaces and vector bundles is that $\mathbf{P}(E) = (E - 0)/\mathbf{C}^\times$. On the other hand, both the real and holomorphic tangent bundles of a complex manifold Z will generally be denoted by TZ except in cases where this might be likely to cause confusion.

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2 Complex Contact Manifolds

Definition 2.1 *A complex contact manifold is a pair (Z, D), where Z is a complex manifold and $D \subset TZ = T^{1,0}Z$ is a codimension-1 holomorphic sub-bundle which is maximally non-integrable in the sense that the O'Neill tensor ('Frobenius obstruction')*

$$\begin{array}{rcl} D \times D & \to & TZ/D \\ (v, w) & \mapsto & [v, w] \bmod D \end{array}$$

is everywhere non-degenerate.

The condition of non-integrability has a very useful reformulation, which we shall now describe. Given a codimension-1 holomorphic sub-bundle $D \subset TZ$, let $L := TZ/D$ denote the quotient line bundle; we thus have an exact sequence

$$0 \to D \longrightarrow TZ \xrightarrow{\theta} L \to 0 \tag{2.1}$$

where θ is the tautological projection. But we may also think of θ as a line-bundle-valued 1-form

$$\theta \in \Gamma(Z, \Omega^1(L)) \ ,$$

and so attempt to form its exterior derivative $d\theta$. Unfortunately, this ostensibly depends on a choice of local trivialization; for if ϑ is any 1-form, $d(f\vartheta) = fd\vartheta + df \wedge \vartheta$. However, it is now clear that $d\theta|_D$ *is* well defined as a section of $L \otimes \wedge^2 D^*$, and an elementary computation, which we leave to the reader, shows that $d\theta|_D$, thought of in this way, is exactly the O'Neill tensor mentioned above. Now if the skew form $d\theta|_D$ is to be non-degenerate, D must have positive even rank $2n$, so that Z must have odd complex dimension $2n + 1 \geq 3$. Moreover, the non-degeneracy exactly requires that

$$\theta \wedge (d\theta)^{\wedge n} \neq 0 \tag{2.2}$$

as a section of $\Omega^{2n+1}(L^{n+1})$. This now provides a bundle isomorphism between $L^{\otimes(n+1)}$ and the anti-canonical line bundle $K^{-1} = \wedge^{2n+1}T^{1,0}Z$, in keeping with the isomorphism $D \cong L \otimes D^*$ induced by the O'Neill tensor.

Conversely, let Z be a compact complex $(2n+1)$-manifold with $H^1(Z, \mathbf{Z}_{n+1}) = 0$, and suppose that $c_1(Z)$ is divisible by $n + 1$. Then there is a unique holomorphic line bundle $L := K^{-1/(n+1)}$ such that $L^{\otimes(n+1)} \cong K^{-1}$. If we are then given a twisted holomorphic 1-form

$$\theta \in \Gamma(Z, \Omega^1(K^{-1/(n+1)}))$$

we may then construct

$$\theta \wedge (d\theta)^{\wedge n} \in \Gamma(Z, \Omega^{2n+1}(K^{-1})) = \Gamma(Z, \mathcal{O}) = \mathbf{C} \ .$$

If this constant is non-zero, one calls θ a *complex contact form* because the corresponding $D = \ker \theta$ is then a complex contact structure. Now the holomorphic map

$$\Gamma(Z, \Omega^1(L)) \ni \theta \mapsto \theta \wedge (d\theta)^n \in \mathbf{C}$$

is homogeneous of degree $(n+1)$, and so is represented by a homogeneous polynomial. Since two elements of $\Gamma(Z, \Omega^1(L))$ define the same sub-bundle D iff they are proportional, it follows that the space of contact structures, if non-empty, is the complement of a degree-$(n+1)$ complex hypersurface $S \subset \mathbf{P}[\Gamma(\Omega^1(L))]$; in particular, *the space of all contact structures on Z is a connected complex manifold.*

Example 2.1. Let $\mathbf{V} \cong \mathbf{C}^{2n+2}$ be an even-dimensional complex vector space, and let $\Upsilon \in \wedge^2\mathbf{V}^*$ be any skew form on $\mathbf{V}$. Then there is an associated $\theta \in \Gamma(\mathbf{P}(\mathbf{V}), \Omega^1(2))$ defined by $\varpi^*\theta := \Xi \lrcorner \Upsilon$, where $\varpi : \mathbf{V} - 0 \to \mathbf{P}(\mathbf{V})$ is the canonical projection, and where Ξ is the (Euler) vector field which generates the $\mathbf{C}^\times$-action of scalar multiplication on $\mathbf{V}$; conversely, every element of $\Gamma(\mathbf{P}(\mathbf{V}), \Omega^1(2))$ arises this way from a Υ, as may, for example, be read off from the Euler exact sequence

$$0 \to \Omega^1(2) \to \mathbf{V}^*(1) \to \mathcal{O}(2) \to 0$$

on $\mathbf{P}(\mathbf{V})$. On the other hand, the anti-canonical line bundle K^{-1} of $\mathbf{P}(\mathbf{V})$ is isomorphic to $\mathcal{O}(2n+2)$, so we have $K^{-1/(n+1)} \cong \mathcal{O}(2)$, and any contact structure on $\mathbf{P}(\mathbf{V}) \cong \mathbf{CP}_{2n+1}$ must arise from some $\theta \in \Gamma(\mathbf{P}(\mathbf{V}), \Omega^1(2))$. Now the condition that $\theta \wedge (d\theta)^n \neq 0$ may now be rewritten as $\Upsilon^{n+1} \neq 0$ as a consequence of the observation that $d(\varpi^*\theta) = d(\Xi \lrcorner \Upsilon) = \mathcal{L}_{\Xi} \Upsilon = 2\Upsilon$. Thus the set of contact structures on $\mathbf{CP}_{2n+1}$ is exactly parameterized by the symplectic forms on $\mathbf{C}^{2n+2}$ modulo rescalings. $\diamondsuit$

Example 2.2. Let Y_{n+1} be any complex manifold, and let $Z_{2n+1} = \mathbf{P}(T^*Y)$ be its projectived holomorphic cotangent bundle. Observe that the cotangent bundle T^*Y carries a tautological 1-form $\hat{\theta}$ defined by $\hat{\theta}|_\vartheta = p^*\vartheta$, where $p : T^*Y \to Y$ is the canonical projection. If $\Phi_t : T^*Y \to T^*Y$ is fiber-wise scalar multiplication by $t \in \mathbf{C}^\times$, we also have $\Phi_t^*\hat{\theta} = t\hat{\theta}$, so there is a line-bundle-valued form $\theta \in \Gamma(\mathbf{P}(T^*Y), \Omega^1(L))$ on Z such that $\varpi^*\theta = \hat{\theta}$; here ϖ is the canonical projection from $T^*Y - 0_Y$ to $Z = \mathbf{P}(T^*Y)$, and $L \to Z$ is the holomorphic line bundle whose local sections are homogeneity-1 functions on $T^*Y - 0_Y$. On the other hand, $\Upsilon = d\hat{\theta}$ is a holomorphic symplectic form on T^*Y, and the non-degeneracy of Υ implies that θ is a contact form. In particular, L^{n+1} is isomorphic to the anti-canonical line bundle of $Z = \mathbf{P}(T^*Y)$. $\diamondsuit$

The usual proof of the Darboux theorem [1] for real contact manifolds applies equally well in the complex case; thus, any complex contact manifold (Z, D) of dimension $2n+1$ is locally isomorphic to $(\mathbf{C}^{2n+1}, \ker(dz^{2n+1} + \sum_{j=1}^n z^j dz^{j+n}))$.

As a consequence, any complex contact manifold may be obtained by gluing together open sets in $\mathbf{C}^{2n+1}$ with transitions functions which are *complex contact transformations* — i.e. biholomorphisms which preserve the fixed complex contact structure.

In order to better understand this notion of complex contact transformation, we should first try to understand the infinitesimal version; that is, when does the pseudo-group of biholomorphisms generated by a holomorphic vector field preserve a given contact structure D? The answer is given by the following result:

Proposition 2.1 *Let (Z, D) be any complex contact manifold, and let $\mathbf{u} \in \Gamma(Z, \mathcal{O}(L))$ be any holomorphic section of the contact line bundle. Then there is a unique holomorphic vector field ζ on Z such that $\theta(\zeta) = \mathbf{u}$ and such that the pseudo-group of local transformations of Z generated by ζ consists of complex contact transformations.*

Proof. It suffices to prove the result locally, since the uniqueness will guarantee that local choices of ζ agree on overlaps; thus we may assume that the contact line bundle L is trivial. Relative to a trivialization of L, the contact form θ becomes a holomorphic 1-form ϑ, whereas the section $\mathbf{u}$ is represented by an ordinary holomorphic function u. Now there is a unique holomorphic vector field η such that $\eta \lrcorner d\vartheta = 0$ and $\eta \lrcorner \vartheta = 1$ because $\vartheta \wedge (d\vartheta)^n \neq 0$ and the condition that $\theta(\zeta) = \mathbf{u}$ now becomes $\zeta = u\eta + \xi$ for some $\xi \in \Gamma(\mathcal{O}(D))$. On the other hand, the condition that ζ generate contact transformations can be written as $\pounds_\zeta \vartheta = f\vartheta$ for some holomorphic function f, and we therefore must have

$$(u\eta + \xi) \lrcorner d\vartheta + d[(u\eta + \xi) \lrcorner \vartheta] \equiv 0 \bmod \vartheta \ ,$$

and hence $\xi \lrcorner d\vartheta = -du \bmod \vartheta$. Thus

$$\zeta = u\eta - (d\vartheta|_D)^{-1}(du|_D) \tag{2.3}$$

is the unique infinitesimal contact transformation with $\theta(\zeta) = \mathbf{u}$. ∎

Corollary 2.1 *The exact sequence*

$$0 \to \mathcal{O}(D) \to \mathcal{O}(TZ) \xrightarrow{\theta} \mathcal{O}(L) \to 0 \tag{2.4}$$

splits as a sequence of sheaves of abelian groups. *In particular,*

$$H^p(Z, \mathcal{O}(TZ)) \cong H^p(Z, \mathcal{O}(D)) \oplus H^p(Z, \mathcal{O}(L))$$

for all p.

Remark 2.1. The canonical splitting $\mathcal{O}(TZ) \leftarrow \mathcal{O}(L)$ of (2.4) given by Proposition 2.1 is *not* a homomorphism of $\mathcal{O}$-modules, since (2.3) manifestly involves the first derivative of $\mathbf{u}$. In particular, Corollary 2.1 does *not* assert the existence of a splitting of the associated exact sequence (2.1) of vector bundles. In fact, we will see in Corollary 2.2 that the latter never splits if Z is Fano. $\square$

Proposition 2.2 *Let (Z, D) be a compact complex contact $(2n+1)$-manifold with $H^1(Z, \mathbf{Z}_{n+1}) = 0$. Then Z admits a second contact structure $\tilde{D} \neq D$ iff $\Gamma(Z, \mathcal{O}(D)) \neq 0$.*

Proof. If $\Gamma(Z, \mathcal{O}(D)) \neq 0$, there is a non-trivial holomorphic vector field ζ on Z with $\theta(\zeta) = 0$. Since we also have $\theta(0) = 0$, Proposition 2.1 asserts that this ζ is not an infinitesimal contact transformation. Thus the contact structure $(\exp t\zeta)_* D$ will differ from D if t is sufficiently small.

Conversely, if there is a second contact structure $\tilde{D} \neq D$ on Z, we must have $\tilde{D} = \ker \tilde{\theta}$, $\tilde{\theta} \in \Gamma(Z, \Omega^1(L))$, by virtue of the assumption that $H^1(Z, \mathbf{Z}_{n+1}) = 0$; and since $\tilde{D} \neq D$, there must be a point of Z at which the values of θ and $\tilde{\theta}$ are linearly independent. Thus $\tilde{\theta}|_D \in \Gamma(Z, \mathcal{O}(D^* \otimes L))$ is not zero, and this of course implies $(d\theta|_D)^{-1}(\tilde{\theta}|_D) \in \Gamma(Z, \mathcal{O}(D))$ is not zero, either. $\blacksquare$

Because complex contact structures are locally trivial, and infinitesimal contact automorphisms are specified by holomorphic sections of the contact line bundle L, the space of infinitesimal deformations of a compact complex manifold (Z, D) is exactly $H^1(Z, \mathcal{O}(L))$. Thus the existence of the natural splitting $H^1(Z, \mathcal{O}(TZ)) \leftarrow H^1(Z, \mathcal{O}(L))$ of Corollary 2.1 may be interpreted as saying that *you can't deform the contact structure without deforming the complex structure*. The following result [18, 22] is therefore an immediate consequence of the connectedness of the space of contact forms:

Proposition 2.3 *Let Z be a simply connected compact complex manifold. Then any two complex contact structures on Z are equivalent via some biholomorphism of Z.*

Remark 2.2. There is no real analogue of this; for example, the 3-sphere S^3 carries [5] many inequivalent real contact structures. The essential difference is that the set of real contact forms is typically disconnected. $\square$

Examples 2.1 and 2.2 were both displayed in a way that related them to symplectic structures on $\mathbf{C}^\times$-bundles over the given manifold Z. As you might expect, this is a manifestation of a general procedure [1, Appendix 4E] known as *symplectification*, which we now review. Let $L^{*\times}$ denote the complement of the zero section in in the total space of $L^* = L^{-1}$, and let $\varpi : L^{*\times} \to Z$ denote the canonical projection, which we will consider to be a holomorphic

principal $\mathbf{C}^\times$-bundle. Then $\varpi^* L$ is canonically trivial, although the canonical non-zero section has homogeneity 1 with respect to the $\mathbf{C}^\times$-action. It follows that $\varpi^*\theta$ may be considered to be a holomorphic 1-form on $L^{*\times}$ rather than just a section of $\Omega^1(\varpi^* L)$. We may therefore define $\Upsilon = d(\varpi^*\theta)$, and observe that the non-degeneracy (2.2) of θ is exactly equivalent to requiring that the closed holomorphic 2-form Υ satisfy $\Upsilon^{n+1} \neq 0$; that is, Υ is a *holomorphic symplectic form*, and in particular the map $v \to \Upsilon(v, \cdot)$ is an isomorphism $TL^{*\times} \to T^* L^{*\times}$. Conversely, any symplectic form of homogeneity 1 on a principal $\mathbf{C}^\times$-bundle arises from a complex contact structure on the base by the formula $\varpi^*\theta = \Upsilon(\xi, \cdot)$, where ξ is the vector field which generates the $\mathbf{C}^\times$-action.

This has interesting consequences back down on the contact manifold Z.

Proposition 2.4 *Let (Z, D) be a complex contact manifold, and let $L = TZ/D$ be its contact line bundle. Then the contact form $\theta \in \Gamma(Z, \Omega^1(L))$ gives rise to a non-degenerate section $\Upsilon \in \Gamma(Z, L^* \otimes \wedge^2 J^1 L)$, where $J^1 L$ is the 1-jet bundle of L. In particular, there is an isomorphism $(J^1 L)^* \to L^* \otimes J^1 L$ induced by contraction with Υ. Moreover, this is compatible with the map $D \otimes L^* \to D^*$ induced by $d\theta|_D$.*

Proof. Since a local holomorphic section f of L may be identified with a function $\hat{f}$ on $L^{*\times}$ which has homogeneity 1 with respect to the $\mathbf{C}^\times$-action, the identification $J^1(f) \leftrightarrow d\hat{f}$ gives us a $\mathbf{C}^\times$-invariant identification of $\varpi^* L^* \otimes J^1 L$ with the cotangent bundle of $L^{*\times}$. Thus the symplectic form $\Upsilon = d(\varpi^*\theta)$ on $L^{*\times}$, which transforms under the $\mathbf{C}^\times$-action with homogeneity 1, may be identified with a non-degenerate holomorphic section of $L \otimes \wedge^2(L^* \otimes J^1 L) = L^* \otimes \wedge^2 J^1 L$. Moreover, the diagram

$$
\begin{array}{ccc}
J^1 L & \xrightarrow{\Upsilon^{-1}} & L \otimes (J^1 L)^* \\
\uparrow & & \downarrow \\
\Omega^1(L) & \longrightarrow & TZ \\
\downarrow & & \uparrow \\
D^* \otimes L & \xrightarrow{(d\theta|_D)^{-1}} & D
\end{array}
$$

commutes as a consequence of the fact that $\Upsilon = d(\varpi^*\theta)$. ∎

Remark 2.3. Proposition 2.1 says that $\Gamma(Z, \mathcal{O}(L))$ can be identified with the set of infinitesimal contact transformations of (Z, D), and so has a Lie algebra structure. In fact, the Lie bracket $\Gamma(\mathcal{O}(L)) \times \Gamma(\mathcal{O}(L)) \to \Gamma(\mathcal{O}(L))$ is exactly given by $[\mathbf{u}, \mathbf{v}] = \Upsilon^{-1}(J^1\mathbf{u}, J^1\mathbf{v})$. In the same way, one can also give $\bigoplus_{m=1}^{\infty} \Gamma(Z, \mathcal{O}(L^m))$ the structure of a graded Lie algebra. □

Proposition 2.5 *Let Z be a compact complex $(2n+1)$-manifold of Kähler type, and let D be a compact complex structure on Z. Then the obstruction $\in \mathbf{Ext}^1_Z(\mathcal{O}(L), \mathcal{O}(D)) = H^1(Z, \mathcal{O}(L^* \otimes D))$ to splitting the exact sequence*

$$0 \to D \to TZ \to L \to 0$$

is obtained by applying the composition

$$H^1(\Omega^1) \to H^1(\mathcal{O}(D^*)) \stackrel{(d\theta|_D)^{-1}}{\longrightarrow} H^1(\mathcal{O}(L^* \otimes D))$$

to $\frac{2\pi i}{n+1} c_1(Z) \in H^1(Z, \Omega^1)$.

Proof. The obstruction to splitting the jet sequence

$$0 \to \Omega^1(L) \to J^1 L \to L \to 0$$

is [2] the Atiyah obstruction $a(L) \in H^1(Z, \Omega^1)$, and may be expressed in Čech cohomology as $[d \log f_{\alpha\beta}]$, where $\{f_{\alpha\beta}\}$ is a system of transition functions for L. Thus the image of $a(L)$ in $H^1(Z, \mathcal{O}(L^* \otimes D))$ is the extension class of

$$0 \to D^* \otimes L \to (J^1 L)/\mathcal{O} \to L \to 0. \tag{2.5}$$

But contraction with Υ^{-1} converts (2.5) into the exact sequence

$$0 \to D \to TZ \to L \to 0. \tag{2.6}$$

The result thus follows from Proposition 2.4 and the observation [2] that $a(L) = 2\pi i c_1(L)$ for any line bundle on a compact Kähler manifold. ∎

Corollary 2.2 *Suppose that (Z, D) is a compact complex contact manifold such that $c_1(Z) > 0$. Then (2.1) does not split.*

Proof. Because $K \otimes L$ is a negative line bundle, the Kodaira vanishing theorem implies that $H^1(Z, \mathcal{O}(L^*)) = 0$. Thus the restriction map $H^1(Z, \Omega^1) \to H^1(Z, \mathcal{O}(D^*))$ is injective, and the result follows from Proposition 2.5. ∎

While a number of interesting things can be said regarding complex contact manifolds in general, one seems to need strong extra hypotheses before a classification becomes imaginable. One possible route is to limit ones ambitions to the low dimensional cases; cf. [31]. In this article, however, we will instead study manifolds of the following type:

Definition 2.2 *If (Z, D) is a complex contact manifold such that $c_1(Z) > 0$, we will say that (Z, D) is a* Fano contact manifold*.*

Thus, for example, $\mathbf{CP}_{2n+1}$ and $\mathbf{P}(T^*\mathbf{CP}_n)$ proxide two examples of Fano contact manifolds fo dimension $2n+1$. These are both examples of *homogeneous* complex contact manifolds, meaning that their groups of contact transformations act transitively; there is [30] exactly one such object for each simple complex Lie algebra, and every homogeneous complex contact manifold is automatically Fano as a consequence of Proposition 2.1. From our perspective, however, the overwhelming reason to study such objects is provided by the following result [24, 6]:

Theorem 2.1 (Salamon/Bérard-Bergery) *Let (M^{4n}, g) be a quaternion-Kähler manifold of scalar curvature $16n(n+2)$. Then there is a complex contact $2n+1$-manifold (Z, D), called the* twistor space *of (M, g), which admits a Kähler-Einstein metric h of scalar curvature $8(n+1)(2n+1)$ such that*

(i) there is a Riemannian submersion $\wp : Z \to M$ with totally geodesic fibers S^2 of constant curvature 4;

(ii) the horizontal sub-bundle of this submersion is the contact distribution $D \subset TZ$;

(iii) each fiber of $\wp$ is a rational complex curve $\mathbf{CP}_1 \subset Z$, with normal bundle isomorphic to $[\mathcal{O}(1)]^{\oplus 2n}$; and

(iv) there is a free anti-holomorphic involution $\sigma : Z \to Z$ which commutes with $\wp$.

In order to allow for the case $n = 1$ while ensuring that this theorem remains true, we now supplement Definition 1.1 as follows:

Definition 2.3 *An oriented Riemannian manifold (M, g) of dimension 4 is said to be quaternion-Kähler if it is Einstein and has self-dual Weyl curvature.*

Now one might hope that this twistor-theoretic machinery would provide a powerful source of new examples of Fano contact manifolds. However, the only known positive quaternion-Kähler manifolds are symmetric spaces, and their twistor spaces are exactly the homogeneous complex contact manifolds alluded to above. Indeed, the following result [17, 18] gives one reason to wonder whether there are non-homogeeous examples at all:

Theorem 2.2 *There are, up to biholomorphism, only finitely many Fano contact manifolds (Z, D) of any fixed dimension.*

There are also results [11, 23] that show that there are no non-homogeneous examples with $n = 1, 2$. And in all dimensions one has the following result [17, 18]:

Theorem 2.3 *Let Z be a contact Fano manifold. If $b_2(Z) \geq 2$, then $Z \cong \mathbf{P}(T^*\mathbf{CP}_{n+1})$.*

3 Contact Structures and Einstein Metrics

The following well-known observation is apparently due to Calabi [10].

Proposition 3.1 *Let (Z,h) be a Kähler-Einstein manifold of positive scalar curvature and complex dimension m; and let $K^\times \to Z$ be the $\mathbf{C}^\times$-bundle obtained from the canonical line bundle K by deleting the zero section. Then $K^\times$ carries an incomplete Ricci-flat Kähler metric $\mathbf{h}$ such that $\Phi_t^*\mathbf{h} = |t|^{2/(m+1)}\mathbf{h}$, where $\Phi_t : K \to K$ is scalar multplication by $t \in \mathbf{C}^\times$. Moreover, (Z,h) is a Kähler quotient of $(K^\times, \mathbf{h})$ by $S^1 \subset \mathbf{C}^\times$.*

Proof. Define $r : K \to K$ by $r(x) = \frac{1}{2}\|x\|^{2/m+1}$, where $\|\cdot\|$ is the norm on K associated with the Kähler-Einstein metric and m is the complex dimension of Z. We may then define $\mathbf{h}$ to be the Kähler metric on $K^\times$ associated with the positive $(1,1)$-form $\omega_1 := i\partial\bar{\partial}r$.

We may suppose that the Kähler-Einstein metric h is mormalized so that its Ricci tensor is $2(m+1)h$. The pull-back of ω via the canonical projection $\varpi : K^\times \to Z$ is then

$$\varpi^*\omega = \frac{i}{2}\partial\bar{\partial}\log r,$$

so that (Z,ω) is precisely the symplectic quotient of $(K^\times, \omega_1)$ corresponding to $r = 1/2$. On the other hand, there is a tautological holomorphic $(m,0)$-form ϕ on $K^\times$ defined by $\phi|_\varphi = \varpi^*\varphi$, and the relation between r and the norm $\|\cdot\|$ on $K \to Z$ tells us that

$$\varpi^*\omega^m \propto \frac{\phi\wedge\bar{\phi}}{r^{m+1}},$$

where $\propto$ means the two expressions differ by multiplication by a non-zero constant. On the other hand, if Ξ denotes the holomorphic (Euler) vector field which generates the $\mathbf{C}^\times$-action on $K^\times$, then $\Xi\lrcorner\partial\phi = \pounds_\Xi\phi = \phi$, whereas $\Xi\lrcorner(\partial\log r\wedge\phi) = (\Xi\log r)\phi = \phi/(m+1)$; thus $\partial\phi = (m+1)\partial\log r\wedge\phi$, since both sides have type $(m+1,0)$ and $m+1 = \dim_{\mathbf{C}} K^\times$. Hence

$$\begin{aligned}
(\omega_1)^{m+1} &= (i\partial\bar{\partial}r)^{m+1} \\
&= \left(2r\varpi^*\omega + i\frac{\partial r\wedge\bar{\partial}r}{r}\right)^{m+1} \\
&\propto r^{m-1}\varpi^*\omega^m\wedge\partial r\wedge\bar{\partial}r \\
&\propto r^{m-1}\frac{\phi\wedge\bar{\phi}}{r^{m+1}}\wedge\partial r\wedge\bar{\partial}r \\
&= \phi\wedge\bar{\phi}\wedge\partial\log r\wedge\bar{\partial}\log r \\
&\propto \partial\phi\wedge\overline{\partial\phi}.
\end{aligned}$$

Since $\partial\phi$ is a holomorphic form, the logarithm of the volume form of ω_1 is thus pluri-harmonic, and the Kähler metric $\mathbf{h}$ is Ricci-flat. ∎

Descending back to Z, we thus have the following:

Proposition 3.2 *Let Z be a Fano manifold which admits a Kähler-Einstein metric. Then the 1-jet bundle $J^1K^* \to Z$ of the anti-canonical line bundle of Z admits a natural Hermitian-Einstein inner product induced by the Ricci-flat metric $\mathbf{h}$ on $K^\times$. In particular, J^1K^* is quasi-stable — that is, it is a semi-stable direct sum of stable vector bundles.*

Proof. First observe that the pull-back of $K \otimes J^1K^*$ from Z to $K^\times$ is $\mathbf{C}^\times$-equivariantly isomorphic to $T^*K^\times$, since any local holomorphic section f of K^* on Z can be identified with a holomorphic function $\hat{f}$ on $K^\times$ of homogeneity 1, and the value of the 1-jet of f at a point of Z determines and is determined by the value of $d\hat{f}$ at any point of the corresponding fiber of $K^\times$. In other words, a local section of J^1K^* on Z is a holomorphic cotangent field ψ on $K^\times$ which satisfies $\Phi_t^*\psi = t\psi$ for all $t \in \mathbf{C}^\times$. Since the inner product $\langle\cdot,\cdot\rangle$ on covectors determined by $\mathbf{h}$ satisfies $\Phi_t^*\langle\cdot,\cdot\rangle = |t|^{-2/(m+1)}\langle\cdot,\cdot\rangle$, we can now define an inner product $(\cdot,\cdot)$ on $J^1K^* \to Z$ by

$$(\psi, \tilde{\psi}) := r^{-2m/(m+1)}\langle\psi, \tilde{\psi}\rangle.$$

We claim that this inner product is Hermitian-Einstein.

Since this claim is local in character, we may now restrict to an open set in Z over which we have a root $\ell = K^{-1/(m+1)}$ of the the anti-canonical bundle. Since the curvature of the obvious Chern connection on this root is a constant multiple of the Kähler form ω, it is thus sufficient for us to check that the induced inner product on $\ell^{-m} \otimes J^1K^* \cong J^1\ell$ is Hermitian-Einstein. However, a local section of this bundle may be interpreted as a local holomorphic cotangent vector field ψ on $K^\times$ such that $\Phi_t^*\psi = t^{1/(m+1)}\psi$, and the induced inner product on such objects is just $\langle\cdot,\cdot\rangle$. Thus the curvature of $(T^*K^\times, \langle\cdot,\cdot\rangle)$ is just the pull-back of the curvature of $\ell^{-m} \otimes J^1K^*$; in particular, $T^*K^\times$ is flat along the fibers of $K^\times \to Z$. On the other hand, $\varpi^*\omega \equiv \omega_1/2r \bmod (\partial r, \bar{\partial} r)$, so the projection $\varpi : K^\times \to Z$ is conformal in the horizontal directions. Combining these last two observations tells us that the Ricci curvature $F^\alpha_{\beta j\bar{k}}\omega^{j\bar{k}}$ of $\ell^{-m} \otimes J^1K^*$ pulls back to a multiple of the Ricci curvature of $\mathbf{h}$, and so vanishes. Hence the inner product induced on J^1K^* by $\mathbf{h}$ is Hermitian-Einstein, as claimed. The quasi-stability statement now follows from the work of Kobayashi and Lübke [14, 19]. ∎

Remark 3.1. The extension class of

$$0 \to \mathcal{O} \to (K \otimes J^1K^*)^* \to TZ \to 0$$

is just $c_1(Z) \in H^1(Z, \Omega^1) = Ext^1_Z(TZ, \mathcal{O})$, so the stability aspect of the above result should be attributed to Tian [29]. However, see Remark 3.4 below. □

This stability result now allows us to prove the following:

Theorem 3.1 *Let (Z, D) be a compact complex contact manifold, and suppose that h is a Kähler-Einstein metric of positive scalar curvature on Z. Then the Ricci-flat manifold $(K^\times, \mathbf{h})$ of Proposition 3.1 is finitely covered by a hyper-Kähler manifold.*

Proof. Let $m = 2n+1$ be the complex dimension of Z, and let $L = K^{-1/(n+1)}$ be the contact line bundle. Since $J^1K^* = L^n \otimes J^1L$ is Hermitian-Einstein by Proposition 3.1, it follows that $L^* \otimes \wedge^2 J^1L = L \otimes K^2 \otimes \wedge^2 J^1K^*$ is also Hermitian-Einstein. But the symplectification procedure of page 8 gives us a non-degenerate section Υ of $L^* \otimes \wedge^2 J^1 L \subset \mathcal{H}om((J^1L)^*, L^* \otimes J^1L)$, so contraction with $\Upsilon^{-1} \otimes \Upsilon^{-1}$ is an isomorphism

$$L^* \otimes \wedge^2 J^1 L \xrightarrow{\cong} (L^* \otimes \wedge^2 J^1L)^*.$$

Thus the odd Chern classes of $L^* \otimes \wedge^2 J^1 L$ are all 2-torsion, and in particular this bundle has degree 0. This shows that $L^* \otimes \wedge^2 J^1L$ is actually Hermitian-Ricci-flat, meaning that its curvature $\hat{F}$ satisfies $\omega \cdot \hat{F} = 0$.

Given a contact form $\theta \in \Gamma(Z, \Omega^1(L))$, the symplectification procedure just alluded to produces a non-degenerate $\Upsilon \in \Gamma(Z, \mathcal{O}(L^{-1} \otimes \wedge^2 J^1 L))$, and, since $L^{-1} \otimes \wedge^2 J^1 L$ is Hermitian-Ricci-flat, Υ must be parallel. Indeed, if ∇ is the Chern connection of $L^{-1} \otimes \wedge^2 J^1 L$, we have $\nabla^{0,1}\Upsilon = 0$ because Υ is holomorphic, and whereas

$$\begin{aligned}
\int_Z \|\nabla^{1,0}\Upsilon\|^2 \, d\mu &= -\int_Z \langle \Upsilon, i\omega^{j\bar{k}} \nabla_{\bar{k}} \nabla_j \Upsilon \rangle \, d\mu \\
&= \int_Z \langle \Upsilon, i\omega^{j\bar{k}} \hat{F}_{j\bar{k}}(\Upsilon) \rangle \, d\mu \\
&= 0.
\end{aligned}$$

Hence $\nabla\Upsilon = \nabla^{1,0}\Upsilon + \nabla^{0,1}\Upsilon = 0$.

Let $L^{*\times}$ be the complement of the zero section in L^*, and notice that there is a natural covering map $L^{*\times} \to K^\times$ given by $x \to x^{\otimes(m+1)}$, and recall that the symplectification of procedure of page 8 initially displays Υ as a non-degenerate holomorphic 2-form on $L^{*\times}$, in a manner consistent with our identification of sections of J^1L with homogeneity-1 1-forms on $K^\times$. Thus, pulling $\mathbf{h}$ back to $L^{*\times}$ via this covering, Υ becomes a parallel $(2,0)$-form on the Ricci-flat Kähler manifold $(L^{*\times}, \mathbf{h})$. The holonomy group $\subset SU(2n+2)$ of $\mathbf{h}$ thus stabilizes a $(2,0)$-form of maximal rank, and so must be a subgroup of $Sp(n+1)$. Hence $(L^{*\times}, \mathbf{h})$ is a hyper-Kähler manifold. ∎

Remark 3.2. A more direct way of seeing that $L^* \otimes \wedge^2 J^1 L$ is Hermitian-Ricci-flat is to observe that, in terms of the local root ℓ used in the proof of Proposition 3.2, one has $L^* \otimes \wedge^2 J^1 L = \wedge^2 J^1 \ell$. It's curvature is thus explicitly given by $\hat{F}^{\alpha\beta}_{\gamma\epsilon j\bar{k}} = 2\delta^{[\alpha}_{[\gamma} F^{\beta]}_{\epsilon]j\bar{k}}$, and so is annihilated by contraction with $\omega^{j\bar{k}}$. □

Theorem 3.2 *Let (Z, h) be a compact Kähler-Einstein manifold with positive scalar curvature. If Z admits two distinct contact structures D and $\tilde{D}$, then the associated Ricci-flat manifold $(K^{\times}, \mathbf{h})$ is actually flat.*

Proof. The same argument used in the proof of Theorem 3.1 says that the holonomy group of $(L^{*\times}, \mathbf{h})$ now stabilizes *two* linearly independent $(2,0)$-forms of maximal rank; moreover, these two $(2,0)$-forms are homogeneous of degree 1 with respect to the $\mathbf{R}^{+}$-action generated by $\xi = \text{grad} r$. Thus $L^{*\times}$ can then be covered by open sets U which split as Riemannian products $(U_1, \mathbf{h}_1) \times (U_2, \mathbf{h}_2)$ of pairs hyper-Kähler manifolds; moreover, these local splittings fit together to yield an orthogonal pair of foliations of $L^{*\times}$, and these foliations are $\mathbf{R}^{+}$-invariant. But the homogeneity of $\mathbf{h}$ says that $\pounds_{\xi}\mathbf{h} = 2\mathbf{h}$, and its components ξ_1 and ξ_2 tangent to U_1 and U_2 therefore satisfy $\pounds_{\xi_1}\mathbf{h}_1 = 2\mathbf{h}_1$ and $\pounds_{\xi_2}\mathbf{h}_2 = 2\mathbf{h}_2$; that is, ξ, ξ_1, and ξ_2 are *homothetic vector fields.* If we now let $S \subset L^{*\times}$ denote the real hypersurface $S = 1$, then ξ is a unit normal vector field on S, and the homothetic nature of ξ implies that the second fundamental form of S is just the restriction $\mathbf{g}$ of $\mathbf{h}$ to S. If now we defined $f : S \to \mathbf{R}$ to be $|\xi_1|^2$, then f is not identically zero and $|df| = |\xi_1|\,|\xi_2|$; hence ξ_1 and ξ_2 respectively have zeroes at the minima and maxima of f. Now if $p \in S$ is a point at which at which $\xi_1 = 0$, let us consider the diffeomorphism $\exp \xi_1 : U_1' \to U_1$ induced on a small leaf-wise neighborhood $U_1' \subset U_1$ of p. Since $(\exp \xi_1)^*\mathbf{h}_1 = e^2\mathbf{h}_1$, its derivative $(\exp \xi_1)_{*p} : T_pU_1 \to T_pU_1$ at p is diagonalizable over $\mathbf{C}$, with all eigenvalues of modulus $e^2 > 1$. Hence all the eigenvalues of the induced push-forward maps on $T_pU_1 \otimes \bigotimes^{k+3}(T_p^*U_1)$ have modulus $e^{-2(k+2)} < 1$. However, since $(\exp \xi_1)_*$ just multiplies $\mathbf{h}_1$ by a constant, it preserves the covariant derivatives $\nabla^{(k)}\mathcal{R} := \nabla \cdots \nabla \mathcal{R}$ of the curvature tensor of $\mathbf{h}_1$; and since 1 is not an eigenvalue of

$$(\exp \xi_1)_{*p} : T_pU_1 \otimes \bigotimes^{k+3}(T_p^*U_1) \to T_pU_1 \otimes \bigotimes^{k+3}(T_p^*U_1),$$

it follows that $\nabla^{(k)}\mathcal{R}$ vanishes at p for all k. Thus all the U_1 components of the curvature tensor of $\mathbf{h}$ on $U = U_1 \times U_2$ vanish to infinite order at p, and since $\mathbf{h}$ is real-analytic, it follows that all its first-foliation curvature components are identically zero on $L^{*\times}$. Since the same argument can also be applied to ξ_2, we conclude that $\mathbf{h}$ is actually flat. ∎

Theorem B *Let Z be a compact complex $(2n+1)$-manifold which admits a Kähler-Einstein metric of positive scalar curvature. Then Z admits two distinct complex contact structures iff $Z \cong \mathbf{CP}_{2n+1}$.*

Proof. By Theorem 3.1, the Ricci-flat Kähler metric $\mathbf{h}$ is flat, whereas the proof also tells us that the real hypersurface S given by $r = 1/2$ has second fundamental form equal to the induced metric $\mathbf{g}$. By the Gauss-Codazzi equations, $(S, \mathbf{g})$ therefore has constant curvature 1, and so is locally isometric to

the standard unit sphere in $\mathbf{R}^{4n+4}$. If η denotes the vector field which generates the circle action $x \to e^{it}x$ on S, then η is a unit Killing vector field on S, and so corresponds to a $(4n+4)\times(4n+4)$ matrix which is simultaneously in $so(4n+4)$ and $SO(4n+4)$. Such a matrix is an isometric complex structure on $\mathbf{R}^{4n+4}$. We may therefore isometrically identify any sufficiently small open set in S with an open set in the unit sphere in $S^{4n+3} \subset \mathbf{C}^{2n+2}$ in such a way that η corresponds to the generator of multiplication by e^{it}. But the fibration $S \to Z$ is a Riemannian submersion, and the fibers are just the orbits of the $U(1)$-action generated by η. Hence $Z = S/U(1)$ is locally isometric to the Fubini-Study metric on $\mathbf{CP}_{2n+1} = S^{4n+3}/U(1)$. The developing map thus gives us an open isometric immersion from the universal cover of (Z,h) to the symmetric space $\mathbf{CP}_{2n+1}$; and since Z and $\mathbf{CP}_{2n+1}$ are both compact and simply connected, this developing map therefore defines a global isometry between (Z,h) and the symmetric space $\mathbf{CP}_{2n+1}$. Since there is only one complex structure which is compatible with the Fubini-Study metric, this isometry is necessarily a biholomorphism, and we are done. ∎

Remark 3.3. By essentially the same argument, one can also sharpen Proposition 3.2 as follows: *if (Z,h) is a compact Kähler-Einstein m-manifold of positive scalar curvature, and if $J^1K_Z^*$ is not strictly stable, then $Z \cong \mathbf{CP}_m$.* □

With Proposition 2.2, Theorem B now implies

Corollary 3.1 *Suppose that $Z_{2n+1} \not\cong \mathbf{CP}_{2n+1}$ is a Fano manifold with complex contact structure D. Also assume that Z admits a Kähler-Einstein metric h. Then $\Gamma(Z,\mathcal{O}(D)) = 0$.*

Corollary 3.2 *Let (Z,D) be a Fano contact manifold which admits a Kähler-Enstein metric. Then the following are equivalent:*

(a) $Z \cong \mathbf{CP}_{2n+1}$;

(b) $h^0(Z,\Omega^1(L)) > 1$;

(c) Z admits more than one complex contact structure;

(d) Z has an automorphism which is not a contact transformation;

(e) $h^0(Z,\mathcal{O}(D)) > 0$;

(f) $\chi(Z,\mathcal{O}(D)) > 0$.

Proof. For any Fano contact manifold (Z,D) one has $H^p(Z,\mathcal{O}(D)) = 0\ \forall p > 1$ as a consequence [18, 24] of the Kodaira vanishing theorem. Thus, invoking

Theorem B and the proof of Proposition 2.1, (f) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a) $\Rightarrow$ (f). On the other hand, the equivalence of (b) and (c) is obvious. ∎

Remark 3.4. Let Z be a Fano manifold of complex dimension m and Picard number $b_2(Z) = 1$. Tian [29, Theorem 2.1] claims to prove that if the tangent bundle of Z admits a proper holomorphic sub-bundle $D \subset TZ$, then

$$\frac{c_1(D)}{c_1(Z)} \leq \frac{\mathrm{rank} D}{m+1}, \tag{3.1}$$

with equality only if there is a non-trivial holomorphic section of $D \subset TZ$. However, the complex contact structure D of any twistor space Z_{2n+1} is a sub-bundle with $\frac{c_1(D)}{c_1(Z)} = \frac{n}{n+1} = \frac{\mathrm{rank} D}{m+1}$. Thus, by Corollary 3.1, the twistor space of $\widetilde{Gr}_4(\mathbf{R}^{n+4}) = SO(n+4)/SO(n) \times SO(4)$ provides a counter-example to this assertion, as does the twistor space of any exceptional Wolf space [30]. Fortunately, the mistake seems to occur only in the last line of the proof, where it is implicitly assumed that the contraction of two non-zero tensors is necessarily non-zero. The proof of (3.1) is thus unaffected. □

Theorem A *Let Z be a Fano contact manifold. Then Z is the twistor space of a quaternion-Kähler manifold iff it admits a Kähler-Einstein metric.*

Proof. Suppose that Z is Kähler-Einstein and admits a complex contact structure. By Theorem 3.1, the incomplete Ricci-flat metric $\mathbf{h}$ on $L^{*\times}$ admits a non-degenerate parallel $(2,0)$-form Υ of homogeneity 1, and in particular is hyper-Kähler. Expressing such a form in terms of its real and imaginary parts, we have $\Upsilon = \omega_3 + i\omega_2$, and using the metric $\mathbf{h}$ transform the 2-forms ω_1, ω_2, into skew endomorphisms $\mathbf{J}_2$ and $\mathbf{J}_3$ of $TL^{*\times}$, we have $\mathbf{J}_1\mathbf{J}_2 = -\mathbf{J}_2\mathbf{J}_1 = \mathbf{J}_3$ and $\mathbf{J}_3\mathbf{J}_1 = -\mathbf{J}_1\mathbf{J}_3 = \mathbf{J}_2$, where $\mathbf{J}_1$ denotes the tautological complex structure of $L^{*\times}$. But since $\mathbf{J}_2 \neq 0$ is a parallel infinitesimal orthogonal transformation of $TL^{*\times}$, $\mathbf{J}_2^2$ has a negative eigenvalue, and by replacing Υ with a suitable real multiple, we may assume that this eigenvalue is -1. The (-1)-eigenspaces of $\mathbf{J}_2^2$ now form a parallel sub-bundle of $L^{*\times}$ which is invariant under the $\mathbf{R}^+$-action generated by ξ. If this is a proper-sub-bundle, the proofs of Theorems 3.2 and B then show that Z is the twistor space $\mathbf{CP}_{2n+1}$ of $\mathbf{HP}_n$. Otherwise, $\mathbf{J}_2^2 = -1$ and hence $\mathbf{J}_3^2 = -1$, so that $\mathbf{J}_A$, $A = 1, 2, 3$, is an anti-commuting triple of compatible complex structures for our hyper-Kähler metric $\mathbf{h}$, and the corresponding Kähler forms ω_A satisfy $\pounds_\xi \omega_A = 2\omega_A$, $A = 1, 2, 3$, as a consequence of the homogeneity of Υ ($A = 2, 3$) and our definition of $\mathbf{h}$ ($A = 1$). But since the 2-forms ω_A are closed, this may be rewritten as $d(\xi \lrcorner \omega_A) = 2\omega_A$; and as $\xi \lrcorner \omega_A = \mathbf{h}(\mathbf{J}_A \mathrm{grad} r, \cdot) = \mathbf{J}_A dr$, we thus have

$$i\partial_A \bar{\partial}_A r = \tfrac{1}{2} d\mathbf{J}_A dr = \omega_A, \;\; A = 1, 2, 3.$$

But these three equations exactly say that r is a *hyper-Kähler potential* in the sense of Swann [27], and [27, Proposition 5.5] now tells us that the $\mathbf{J}_A\xi$ are normalized generators of an isometric $sp(1)$-action on the hypersurface $S \subset L^{*\times}$ defined by $r = 1/2$. Since S is compact, this action exponentiates to yield $Sp(1)$-action, and since, by construction, the normalized generator $\mathbf{J}_1\xi$ has period π, this action descends to an action of $SO(3) = Sp(1)/\mathbf{Z}_2$. And since, by construction, the $SO(2)$-action generated by $\mathbf{J}_1\xi$ is free, this $SO(3)$-action is free, as a consequence of the transitivity of the adjoint action of $SO(3)$ on the unit sphere in $so(3)$. Thus $M := S/SO(3) = L^{*\times}/\mathbf{H}^\times$ is a smooth compact $4n$-manifold, and the Riemannian submersion metric g on M is [27, Theorem 5.1] quaternion-Kähler, with twistor space Z. ∎

Remark 3.5. Instead of invoking Swann's results [27] on the relation between hyper-Kähler and quaternion-Kähler manifolds, we could have instead proceeded by showing that the triple of vector fields $\mathbf{J}_A\xi$ give the hypersurface S a so-called *3-Sasakian structure* [12, 15]. Long relegated to obscurity and neglect, this concept has recently turned out to be a remarkably fruitful source of compact Einstein manifolds [9]. Theorem B is similarly related to [28]. □

Remark 3.6. Uwe Semmelmann has brought some related results [13, 21] to our attention which, when Z is Kähler-Einstein and *spin*, link the presence of a complex contact structure to that of a *Killing spinor*; when Z is spin, this would provide yet another strategy for proving Theorem A. Note, however, that a Fano contact $2n+1$-manifold $Z \neq \mathbf{CP}_{2n+1}$ is spin iff n is odd; thus this method would only prove 'half' of the result in question. □

The following result is well-known; cf. [3, 8, 16, 22].

Corollary 3.3 *Suppose that the twistor space Z of the positive quaternion-Kähler manifold (M, g) is biholomorphic to $\mathbf{CP}_{2n+1}$. Then $(M, g) \cong \mathbf{HP}_n$.*

Proof. Since $h^0(Z, \Omega^1(K^{-1/(n+1)})) = h^0(\mathbf{CP}_{2n+1}, \Omega^1(2)) = (n+1)(2n+1) > 1$, Z has more than one complex contact structure. The Swann hyper-Kähler metric $\mathbf{h}$ associated with Salamon's Kähler-Einstein metric h on Z is therefore flat by Theorem 3.2. The totally umbilic hypersurface given by $r = 1/2$ is therefore locally isometric to the unit sphere in $\mathbf{R}^{4n+4}$, and the three unit Killing fields $\mathbf{J}_A\xi$ define three orthogonal complex structures on this $\mathbf{R}^{4n+4}$ which generate a representation of $sp(1)$. Hence S is locally isometric to $S^{4n+3} \subset \mathbf{H}^{n+1}$ in a manner which sends these three Killing fields to right multiplication by i, j, and k. Thus M is locally isometric to $\mathbf{HP}_n = S^{4n+3}/Sp(1)$, and the developing map construction therefore produces a global isometry $M \to \mathbf{HP}_n$. ∎

Theorem B now yields a much better proof of [18, Theorem 3.2]:

Corollary 3.4 *Let (M,g) and $(\tilde{M},\tilde{g})$ be two positive quaternion-Kähler manifolds, and let Z and $\tilde{Z}$ be their respective twistor spaces. Then $(M,g)\cong(\tilde{M},\tilde{g})$ as Riemannian manifolds iff $Z\cong\tilde{Z}$ as complex manifolds.*

Proof. By Corollary 3.3, we may assume henceforth that $Z,\tilde{Z}\not\cong\mathbf{CP}_{2n+1}$. Theorem B thus tells us that the only complex contact structures on Z and $\tilde{Z}$, respectively, are the horizontal distributions of the twistor projections $\wp : Z\to M$ and $\tilde{\wp} : \tilde{Z}\to\tilde{M}$.

Now suppose that Z and $\tilde{Z}$ are biholomorphic. The Bando-Mabuchi uniqueness theorem for Kähler-Einstein metrics [4] then implies that that there is a biholomorphic *isometry* Φ between the the Kähler-Einstein manifolds (Z,h) and $(Z,\tilde{h})$. But because Z and $\tilde{Z}$ have only one complex contact structure apiece, the isometry Φ sends horizontal subspaces of $\wp$ to horizontal subspaces of $\tilde{\wp}$, and hence fibers of $\wp$ to fibers of $\tilde{\wp}$. Thus $\Phi : Z\to\tilde{Z}$ covers a diffeomorphism $F : M\to\tilde{M}$. But since the twistor projections are Riemannian submersions, F is automatically an isometry, and $(M,g)\cong(M,\tilde{g})$.

The converse follows from the naturality of the Salamon construction. ∎

Our next application concerns the following:

Definition 3.1 *Let (M,g) be a quaternion-Kähler $4n$-manifold, and suppose that $F : M\to M$ is a diffeomorphism. We will say that F is a* quaternionic automorphism *if the derivative of F preserves the $[GL(n,\mathbf{H})\times Sp(1)]/\mathbf{Z}_2$ determined by the quaternion-Kähler structure.*

Corollary C *If $(M,g)\not\cong\mathbf{HP}_n$ is a positive quaternion-Kähler $4n$-manifold, any quaternionic automorphism of M is an isometry.*

Proof. Since the construction of the complex manifold Z only involves [25, 3] the $GL(n,\mathbf{H})Sp(1)$-structure of M, any quaternionic automorphism $F : M\to M$ lifts uniquely to a biholomorphism $\Phi : Z\to Z$, and this biholomorphism will be contact iff [22, 16] F is an isometry. But if Φ is *not* contact, Φ_*D is a second contact structure, and the result follows from Theorem B and Corollary 3.3. ∎

Another interesting consequence concerns the spectrum of the Laplacian:

Corollary 3.5 *Let (M,g) be a compact quaternion-Kähler $4n$-manifold of scalar curvature s. Let λ_1 denote the smallest positive eigenvalue of the Laplace operator $\Delta : C^\infty(M)\to C^\infty(M)$. Then*

$$\lambda_1\geq\frac{(n+1)s}{2n(n+2)}\ ,$$

with equality iff M is $\mathbf{HP}_n$ and g is a multiple of the standard metric.

Proof. Since the statement is empty if $s \leq 0$, we may assume that $s > 0$; and by scale invariance, we may therefore assume henceforth that g has scalar curvature $16n(n+2)$. In conjunction with our usual convention that the Kähler-Einstein metric h on Z has Ricci tensor $4(n+1)h$, this makes the twistor projection $\wp : Z \to M$ into a Riemannian submersion with totally geodesic fibers. If f is an eigenfunction of Δ on M, with eigenvalue $\lambda > 0$, $\hat{f} = \wp^* f$ is an eigenfunction of Δ on Z, also with the eigenvalue λ; moreover, grad$\hat{f}$ is perpendicular to the fibers of $\wp$, and so is a section of D. But $\Delta(\Delta - 8(n+1)) = 4\nabla_\nu \nabla^{\bar{\mu}} \nabla_{\bar{\mu}} \nabla^\nu$ on Z, and the latter elliptic operator is manifestly non-negative; since $\hat{f}$ is an eigenfunction of this operator, with eigenvalue $\lambda(\lambda - 8(n+1))$, it follows that $\lambda \geq 8(n+1)$. Moreover [20], $\hat{f}$ is in the kernel of this operator only if grad$\hat{f}$ is a real-holomorphic vector field, so that $\lambda = 8(n+1)$ only if D admits a non-trivial holomorphic section. Thus, by Corollary 3.1, $\lambda > 8(n+1)$ unless $M \cong \mathbf{HP}_n$. Conversely, $8(n+1)$ actually *is* in the spectrum of $\mathbf{HP}_n$, since all of the many holomorphic sections of $D \subset T\mathbf{CP}_{2n+1}$ are [20] of the form grad$\hat{f}_1 + J$grad$\hat{f}_2$, where $\hat{f}_1$ and $\hat{f}_2$ are of eigenfunctions of Δ with eigenvalue $8(n+1)$. ∎

References

[1] V.I. Arnold, **Mathematical Methods in Classical Mechanics**, Springer, 1989.

[2] M.F. Atiyah, "Complex Analytic Connections in Fibre Bundles," **Trans. Am. Math. Soc. 85** (1957) 181-207.

[3] T.N. Baily and M.G. Eastwood, "Complex Paraconformal Manifolds—Their Differential Geometry and Twistor Theory," **Forum Math. 3** (1991) 61-103.

[4] S. Bando and T. Mabuchi, "Uniqueness of Kähler-Einstein Metrics Modulo Connected Group Actions," **Algebraic Geometry, Sendai, 1985**, (T. Oda, ed), North-Holland 1985.

[5] D. Bennequin, "Enlacements en Equations de Pfaff," **Asterisque 107** (1982) 87-162.

[6] L. Bérard-Bergery, "Variétés Quaternioniennes," unpublished lecture notes, Espalion, 1979.

[7] M. Berger, "Sur les Groupes d'holonomie des Variétés à Connexion Affine et des Variétiés Riemanniennes," **Bull. Soc. Math. France 83** (1955) 278-330

[8] A. L. Besse, **Einstein Manifolds**, Springer-Verlag, 1986.

[9] C.P. Boyer, K. Galicki, and B.M. Mann, "The Geometry and Topology of 3-Sasakian Manifolds," preprint, 1993.

[10] E. Calabi, "Métriques Kählériennes et Fibrés Holomorphes," **Ann. Sci. ENS 12** (1979) 268-294.

[11] N.J. Hitchin, "Kählerian Twistor Spaces," **Proc. Lond. Math. Soc. 43** (1981) 133–150.

[12] S. Ishihara, "Quaternion Kähler Manifolds and Fibered Riemannian Spaces with Sasakian 3-Structure," **Kodai Math. Sem. Rep. 25** (1973) 321-329.

[13] K.-D. Kirchberg and U. Semmelmann, "Complex Contact Structures and the First Eigenvalue of the Dirac Operator on Kähler Manifolds," preprint, Humbolt Universität zu Berlin, 1994.

[14] S. Kobayashi, "Curvature and Stability of Vector Bundles," **Proc. Japan Acad. 58** (1982) 158-162.

[15] M. Konishi, "On Manifolds with Sasakian 3-Structure over Quaternion Kähler Manifolds," **Kodai Math. Sem. 26** (1975) 194-200.

[16] C. LeBrun, "Quaternionic-Kähler Manifolds and Conformal Geometry," **Math. Ann. 284** (1989) 353-376.

[17] C. LeBrun, "A Finiteness Theorem for Quaternionic-Kähler Manifolds with Positive Scalar Curvature," in *The Penrose Transform and Analytic Cohomology in Representation Theory*, *Eastwood, Wolf, Zierau, eds.* **Contemp. Maths. 154** (1993) 89-101.

[18] C. LeBrun and S. Salamon, "Strong Rigidity of Quaternion-Kähler Manifolds," **Inv. Math.** *to appear.*

[19] M. Lübke, "Stability of Einstein-Kähler Vector Bundles," **Man. Math. 42** (1983) 245-257.

[20] Y. Matsushima, "Sur la Structure du Groupe d'Homéomorphismes d'une Certaine Varieté Kählérienne," **Nagoya Math. Journ.** 11 (1957) 145-150.

[21] A. Moroianu, "La Première Valeur Propre de l'Operateur de Dirac sur les Variétés Kählériennes Compactes," preprint, 1994.

[22] T. Nitta, M. Takeuchi, "Contact Structures on Twistor Spaces," **J. Math. Soc. Japan 39** (1987) 139-162.

[23] Y.-S. Poon and S.M. Salamon, "Eight-Dimensional Quaternionic-Kähler Manifolds with Positive Scalar Curvature," **J. Diff. Geom. 33** (1991) 363-378.

[24] S.M. Salamon, "Quaternionic-Kähler Manifolds," **Inv. Math. 67** (1982) 143-171.

[25] S.M. Salamon, "Differential Geometry of Quaternionic Manifolds," **Ann. Scient. ENS 19** (1987) 31-55.

[26] S.M. Salamon, **Riemannian Geometry and Holonomy Groups**, Pitman Research Notes in Mathematics 201, Longman Scientific, 1989.

[27] A.F. Swann, "Hyperkähler and Quaternionic Kähler Geometry," **Math. Ann. 289** (1991) 421-450.

[28] S.-I. Tachibana and W.-N. Yu, "On a Riemannian Space Admitting More than One Sasakian Structure," **Tôhoku Math. J. 22** (1970) 536-540.

[29] G. Tian, "On the Stability of Fano Varieties," **Int. J. Math. 3** (1992) 401-413.

[30] J. A. Wolf, "Complex Homogeneous Contact Structures and Quaternionic Symmetric Spaces," **J. Math. Mech. 14** (1965) 1033-1047

[31] Y.-G. Ye, "A Note on Complex Threefolds Admitting a Holomorphic Contact Structure," **Inv. Math. 115** (1994) 311-314.